PERTH MODERN SCHOOL

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INDEPENDENT PUBLIC SCHOOL

WAEP Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One: Calculator-free

SOLUTIONS

Student number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes

Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet

Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

Let $A = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix}$.

Determine

(a) AA^{-1}. (1 mark)

Solution
$AA^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Specific behaviours
✓ identity matrix

(b) $2A + B$. (2 marks)

Solution
$2 \times \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 1 & 6 \end{bmatrix}$
Specific behaviours
✓ multiple of A
✓ correct sum

(c) AB. (2 marks)

Solution
$\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -15 & -12 \end{bmatrix}$
Specific behaviours
✓ at least two correct elements
✓ correct product

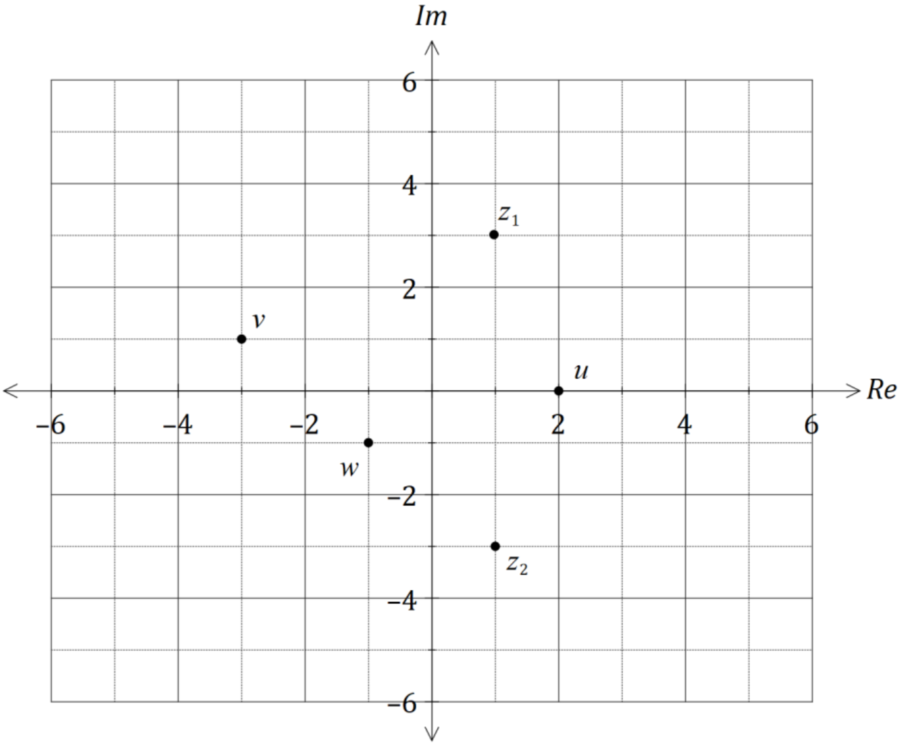
(d) B^{-1}. (2 marks)

Solution
$\begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix}^{-1} = \dfrac{1}{30 - 28} \times \begin{bmatrix} 6 & -4 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3.5 & 2.5 \end{bmatrix}$
Specific behaviours
✓ indicates determinant
✓ correct inverse

Question 2 **(4 marks)**

z_1 and z_2 are the complex solutions to the equation $z^2 - 2z + 10 = 0$.

The location of z_1 is shown in the complex plane below.



Add, with a label, the following complex numbers to the complex plane above.

(a) z_2. (1 mark)

(b) $u = z_1 + z_2$. (1 mark)

(c) $v = iz_1$. (1 mark)

(d) $w = \overline{u + v}$. (1 mark)

Solution
$z_2 = \overline{z_1} = 1 - 3i$ $u = 2$ $v = i(1 + 3i) = -3 + i$ $w = \overline{-1 + i} = -1 - i$
Specific behaviours
✓ plots z_2 ✓ plots u ✓ plots v ✓ plots w

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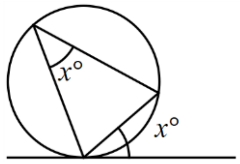
Question 3 **(6 marks)**

(a) Draw a neat diagram to illustrate the angle in the alternate segment theorem. (2 marks)

Solution

Specific behaviours
✓ diagram correctly shows circle, tangent and triangle ✓ clearly marks angles that are equal

(b) Consider the statement 'if a pentagon is regular, then it has sides of equal length'.

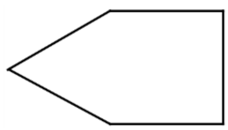
(i) Write the inverse of the statement. (1 mark)

Solution
If a pentagon is **not** regular, then it **does not have** sides of equal length.
Specific behaviours
✓ changes $P \Rightarrow Q$ to $\bar{P} \Rightarrow \bar{Q}$

(ii) Write the contrapositive of the statement. (1 mark)

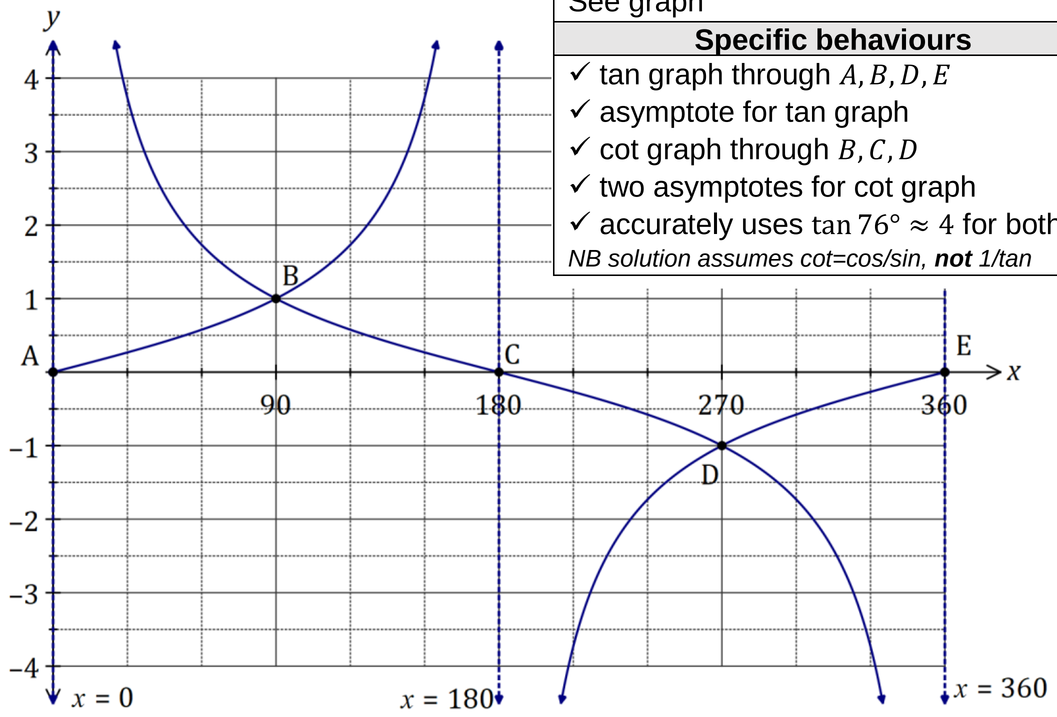
Solution
If a pentagon **does not have** sides of equal length, then it is **not** regular.
Specific behaviours
✓ changes $P \Rightarrow Q$ to $\bar{Q} \Rightarrow \bar{P}$

(iii) Write the converse of the statement and neatly sketch a counter-example to show that the converse is not true. (2 marks)

Solution
If a pentagon has sides of equal length, then it is regular. 
Specific behaviours
✓ reverses $P \Rightarrow Q$ to $Q \Rightarrow P$ ✓ diagram shows irregular pentagon with equal sides

Question 4 **(8 marks)**

(a) On the axes below, sketch the graphs of $y = \tan\left(\frac{x}{2}\right)$ and $y = \cot\left(\frac{x}{2}\right)$, where $0 \le x \le 360°$, labelling all asymptotes. Note that $\tan 76° \approx 4$. (5 marks)



Solution
See graph
Specific behaviours
✓ tan graph through A, B, D, E
✓ asymptote for tan graph
✓ cot graph through B, C, D
✓ two asymptotes for cot graph
✓ accurately uses $\tan 76° \approx 4$ for both
*NB solution assumes cot=cos/sin, **not** 1/tan*

(b) Solve the equation $\sec^2(x) = 3\tan^2(x) - 1$ over the interval $0 \le x \le 360°$. (3 marks)

Solution
$1 + \tan^2(x) = 3\tan^2(x) - 1$ $\tan^2(x) = 1$ $\tan x = \pm 1$ $x = 45°, 135°, 225°, 315°$
Specific behaviours
✓ uses Pythagorean identity ✓ correct values for $\tan x$ ✓ correct solutions

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Question 5 **(6 marks)**

The position vectors of points P and Q are $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively.

(a) Determine the magnitude of the displacement vector $\overrightarrow{PQ}$. (2 marks)

Solution
$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $\lvert\overrightarrow{PQ}\rvert = 2\sqrt{2}$
Specific behaviours
✓ correct $\overrightarrow{PQ}$ ✓ correct magnitude

(b) Determine the values of λ so that $|\mathbf{p} - \lambda\mathbf{q}| = 4$. (4 marks)

Solution
$\mathbf{p} - \lambda\mathbf{q} = \begin{bmatrix} 3 - \lambda \\ 1 + \lambda \end{bmatrix}$ $\therefore (3 - \lambda)^2 + (1 + \lambda)^2 = 16$ $\lambda^2 - 6\lambda + 9 + \lambda^2 + 2\lambda + 1 = 16$ $2\lambda^2 - 4\lambda - 6 = 0$ $\lambda^2 - 2\lambda - 3 = 0$ $(\lambda + 1)(\lambda - 3) = 0$ $\lambda = -1, \quad \lambda = 3$
Specific behaviours
✓ expression for vector in terms of λ ✓ equation for magnitude of vector   ✓ simplifies and factorises equation   ✓ states both values for λ

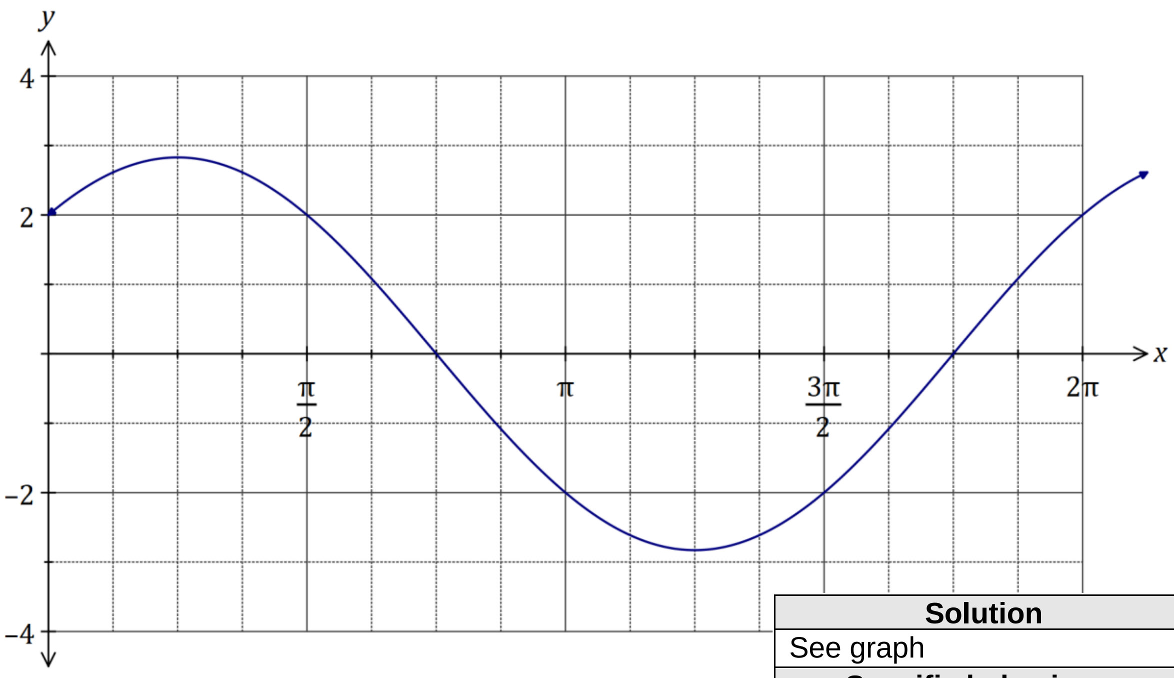
Question 6 **(8 marks)**

Let $f(x) = 2\sin x + 2\cos x$.

(a) Express $f(x)$ in the form $r\sin(x+\alpha)$ where $0 < \alpha < \frac{\pi}{2}$. (4 marks)

Solution
$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ $\sin(x+\alpha) = \frac{2}{2\sqrt{2}}\sin x + \frac{2}{2\sqrt{2}}\cos x$ $= \sin x\cos\alpha + \cos x\sin\alpha$ $\therefore \cos\alpha = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}, \sin\alpha = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4}$ $f(x) = 2\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$
Specific behaviours
✓ value of r ✓ values of $\sin\alpha$ and $\cos\alpha$ ✓ value of α ✓ expression for $f(x)$

(b) Sketch the graph of $y = f(x)$. (4 marks)



Solution
See graph
Specific behaviours
✓ y-intercept   ✓ roots   ✓ maximum and minimum   ✓ smooth sinusoidal curve

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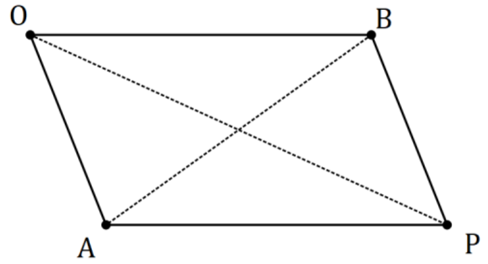
Question 7 **(5 marks)**

Let $OAPB$ be a parallelogram where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



Use a vector method to prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Solution
RTP: $\lvert OP\rvert^2 + \lvert BA\rvert^2 = \lvert OA\rvert^2 + \lvert OB\rvert^2 + \lvert BP\rvert^2 + \lvert AP\rvert^2$ Note: $\lvert\mathbf{r}\rvert^2 = \mathbf{r}\cdot\mathbf{r}$ $LHS = \lvert OP\rvert^2 + \lvert BA\rvert^2$ $= \lvert\mathbf{a}+\mathbf{b}\rvert^2 + \lvert\mathbf{a}-\mathbf{b}\rvert^2$ $= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ $= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}$ $= \lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 + \lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2$ $= \lvert OA\rvert^2 + \lvert OB\rvert^2 + \lvert BP\rvert^2 + \lvert AP\rvert^2$ $= RHS$
Specific behaviours
✓ expresses $\lvert OP\rvert$ and $\lvert BA\rvert$ in terms of $\mathbf{a}$ and $\mathbf{b}$ ✓ uses scalar product to expand sums and differences ✓ simplifies scalar products as magnitudes ✓ expresses in terms of sides ✓ logical presentation of proof

Question 8 **(8 marks)**

(a) Determine all complex solutions to the equation $z^2 - 2z + 5 = 0$. (3 marks)

Solution
$(z-1)^2 - 1 = -5$ $(z-1)^2 = 4i^2$ $z - 1 = \pm 2i$ $z = 1 + 2i, \quad z = 1 - 2i$
Specific behaviours
✓ completes square ✓ simplifies square roots of both sides ✓ states both solutions

(b) Determine the values of the **positive** real constants p and q so that $p - 2i$ is a solution to the equation $2z^2 - qz + 26 = 0$. (5 marks)

Solution
$2(p-2i)^2 - q(p-2i) + 26 = 0$ $2(p^2 - 4pi - 4) - pq + 2qi + 26 = 0$ $2p^2 - 8pi - 8 - pq + 2qi + 26 = 0$ $2p^2 - pq + 18 + (2q - 8p)i = 0$ $\text{Im: } 2q - 8p = 0 \Rightarrow q = 4p$ $\text{Re: } 2p^2 - 4p^2 + 18 = 0 \Rightarrow p^2 = 9$ $p > 0 \Rightarrow p = 3, q = 12$
Specific behaviours
✓ substitutes, expanding square ✓ fully expands and simplifies ✓ uses imaginary terms to express q in terms of p ✓ uses real terms to determine p^2 ✓ states values

End of questions

Supplementary page

Question number: _________